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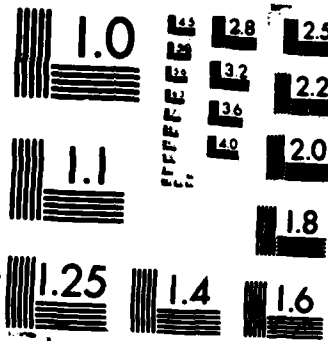HILL CENTER FOR STOCHASTIC PROC   F   MORICZ ET AL.
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Let H be a separable Hilbert space with inner produce denoted by $(\cdot,\cdot)$. By a <u>rowwise orthogonal array</u> we mean a set $\{X_{nk}: k = 1,2,\ldots,n: n = 1,2,\ldots\}$ of random variables (in abbreviation: r.v.'s) such that

(1.1) $\sigma_{nk}^2 := E[||X_{nk}||^2] < \infty$ and

(1.2) $E[(X_{nk}, X_{nk})] = 0$ $(k \neq j; k,j = 1,2,\ldots,n; n = 1,2,\ldots)$.

Details on basic properties and measurability considerations for Hilbert and Banach spaces are available in the literative cited in the references.

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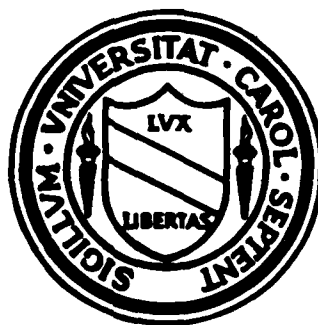
STRONG LAWS OF LARGE NUMBERS FOR ARRAYS OF

ORTHOGONAL RANDOM VARIABLES

BY

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STRONG LAWS OF LARGE NUMBERS FOR ARRAYS OF
ORTHOGONAL RANDOM VARIABLES[1])

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§1. Hilbert Space Valued Random Variables

Let H be a separable Hilbert space with inner product denoted by

$(\cdot,\cdot)$. By a <u>rowwise orthogonal array</u> we mean a set $\{X_{nk}: k = 1,2,\ldots,n;$

$n = 1,2,\ldots\}$ of random variables (in abbreviation: r.v.'s) such that

(1.1) $\sigma^2_{nk}: = E[\|X_{nk}\|^2] < \infty$

and

(1.2) $E[(X_{nk}, X_{nj})] = 0 \quad (k \neq j; \ k,j = 1,2,\ldots,n; \ n = 1,2,\ldots).$

Details on basic properties and measurability considerations for Hilbert

and Banach spaces are available in the literative cited in the references.

We will consider the means

(1.3) $\zeta_n: = \frac{1}{n^\alpha} \sum_{k=1}^{n} X_{nk} \quad (n = 1,2,\ldots)$

---

where α is a fixed positive number. Following Hsu and Robbins (1947) we say that the sequence $\{\zeta_n\}$ __converges to zero completely__, in sign $\lim_{n \to \infty} \zeta_n = 0$ completely, if for every $\epsilon > 0$,

$$\sum_{n=1}^{\infty} P[\,\|\zeta_n\| > \epsilon\,] < \infty.$$

By virtue of the Borel-Cantelli lemma, complete convergence implies __almost sure__ (in abbreviation: a.s.) convergence. The converse is not true in general, except the case when the ζ_n are independent.

First we present a simple sufficient condition to ensure complete convergence.

Theorem 1. __Let $\{X_{nk}\}$ be a rowwise orthogonal array in a separable__ __Hilbert space__ H. __If__

$$(1.4) \qquad \sum_{n=1}^{\infty} \frac{1}{n^{2\alpha}} \sum_{k=1}^{n} \sigma_{nk}^2 < \infty$$

__for some__ $\alpha > 0$, __then__

$$(1.5) \qquad \lim_{n \to \infty} \zeta_n = 0 \ \underline{completely}.$$

Proof. By (1.1) and (1.2),

$$(1.6) \qquad E[\,\|\zeta_n\|^2\,] = \frac{1}{n^{2\alpha}} E\left[\left(\sum_{k=1}^{n} X_{nk}, \ \sum_{j=1}^{n} X_{nj}\right)\right]$$

$$\frac{1}{n^{2\alpha}} \sum_{k=1}^{n} \sum_{j=1}^{n} E[\,(X_{nk}, X_{nj})\,]$$

$$= \frac{1}{n^{2\alpha}} \sum_{k=1}^{n} \sigma_{nk}^2.$$

Hence, by (1.4),

$$\sum_{n=1}^{\infty} E[\|\zeta_n\|^2] = \sum_{n=1}^{\infty} \frac{1}{n^{2\alpha}} \sum_{k=1}^{n} \sigma_{nk}^2 < \infty.$$

This implies (1.5) v he Chebyshev inequality.

A simple conse   e is the following.

Corollary 1. <u>Let</u> $\{X_{nk}\}$ <u>be a rowwise orthogonal array in a separable</u> <u>Hilbert space</u> H <u>such that</u>

$$(1.7) \qquad \sigma_{nk} \leq \sigma_{kk} \qquad (n = k+1, \ k+2, \ \ldots; \ k = 1,2,\ldots).$$

<u>If</u>

$$(1.8) \qquad \sum_{k=1}^{\infty} \frac{\sigma_{kk}^2}{k^{2\alpha-1}} < \infty$$

<u>for some</u> $\alpha > \frac{1}{2}$, <u>then we have</u> (1.5).

Proof. In fact, (1.7) and (1.8) imply (1.4) in case $\alpha > \frac{1}{2}$ as follows

$$\sum_{n=1}^{\infty} \frac{1}{n^{2\alpha}} \sum_{k=1}^{n} \sigma_{nk}^2 \leq \sum_{n=1}^{\infty} \frac{1}{n^{2\alpha}} \sum_{k=1}^{n} \sigma_{kk}^2$$

$$= \sum_{k=1}^{\infty} \sigma_{kk}^2 \sum_{n=k}^{\infty} \frac{1}{n^{2\alpha}} = \mathcal{O}(1) \sum_{k=1}^{\infty} \frac{\sigma_{kk}^2}{k^{2\alpha-1}} < \infty.$$

Remark 1. The weaker condition

$$(1.9) \qquad \sum_{k=1}^{\infty} \frac{\sigma_{kk}^2}{k^{2\alpha}} < \infty$$

implies only the complete convergence of the lacunary subsequence $\{\zeta_{2p}: \ p = 0,1,\ldots\}$. Indeed, by (1.1), (1.2), (1.7) and (1.9),

$$\sum_{p=0}^{\infty} E[\|\zeta_{2^p}\|^2] = \sum_{p=0}^{\infty} \frac{1}{2^{2\alpha p}} \sum_{k=1}^{2^P} \sigma_{2^p,k}^2$$

$$\le \sum_{p=0}^{\infty} \frac{1}{2^{2\alpha p}} \sum_{k=1}^{2^P} \sigma_{kk}^2 = \sum_{k=1}^{\infty} \sigma_{kk}^2 \sum_{p:2^P \ge k} \frac{1}{2^{2\alpha p}}$$

$$= O(1) \sum_{k=1}^{\infty} \frac{\sigma_{kk}^2}{k^{2\alpha}} < \infty.$$

Hence, the Chebyshev inequality yields the complete convergence of $\{\zeta_{2^p}\}$.

Now the surprising fact is that Theorem 1 is the best possible even

(i) for real valued $(H = R)$ r.v.'s; and even

(ii) if we require orthogonality not only within each row but between any two rows in the array $\{X_{nk}\}$.

Theorem 2. <u>Let</u> $\{\sigma_{nk}\}$ <u>be an array of nonnegative numbers such that</u>

$$(1.10) \qquad \sum_{n=1}^{\infty} \frac{1}{n^{2\alpha}} \sum_{k=1}^{n} \sigma_{nk}^2 = \infty$$

<u>for some</u> $\alpha > 0$. <u>Then there exists an array</u> $\{X_{nk}\}$ <u>of real valued r.v.'s</u> <u>such that</u>

$$(1.11) \qquad E[X_{nk}] = 0,$$

$$(1.12) \qquad E[X_{nk}^2] = \sigma_{nk}^2,$$

$$(1.13) \qquad E[X_{nk}X_{mj}] = 0 \qquad (n \neq m \ \underline{or} \ k \neq j;$$

$$k = 1,2,\ldots,n; \ j = 1,2,\ldots,m; \ n,m = 1,2,\ldots),$$

and

$$(1.14) \qquad \limsup_{n \to \infty} |\zeta_n| = \infty \qquad \underline{\text{pointwise}}.$$

Proof. In the counterexample we will construct, the underlying probability space is the unit square $I^2 = [0,1) \times [0,1)$ with the Borel measurable subsets and Lebesgue measure.

By (1.10), there exists a sequence $\{\varepsilon_n\}$ of positive numbers tending to zero such that

$$(1.15) \qquad c_n := \frac{\varepsilon_n^2}{n^{2\alpha}} \sum_{j=1}^{n} \sigma_{nj}^2 \leq 1 \qquad (n = 1,2,\ldots)$$

and

$$(1.16) \qquad \sum_{n=1}^{\infty} c_n = \infty.$$

We define

$$d_{nk} := \sum_{m=1}^{n-1} c_m + \frac{\varepsilon_n^2}{n^{2\alpha}} \sum_{j=1}^{k} \sigma_{nj}^2 \qquad (k = 1,2,\ldots,n; \quad n = 1,2,\ldots)$$

where we mean $\sum_{m=1}^{0} = 0$ in case $n=1$. Denote by $[\cdot]$ the greatest integer part. Define the function $f_{nk}(\omega_1)$ by

$$(1.17) \qquad f_{nk}(\omega_1) := \frac{n^{\alpha}}{\varepsilon_n} I_{J_{nk}}(\omega_1), \qquad \omega_1 \in [0, 1)$$

where $I_{J_{nk}}$ means the indicator of the set J_{nk} and where $J_{nk} = [d_{n,k-1} - [d_{n,k-1}], d_{n,k} - [d_{n,k}])$ when $[d_{n,k-1}] = [d_{n,k}]$ and $J_{nk} = [0, d_{nk}-[d_{nk}]) \cup [d_{n,k-1} - [d_{n,k-1}],1)$ when $[d_{n,k-1}] \neq [d_{n,k}]$. By (1.15) and (1.17),

$$(1.18) \qquad E[f_{nk}^2] = \int_0^1 f_{nk}^2 (\omega_1) = \sigma_{nk}^2 \ ,$$

$$(1.19) \qquad E[f_{nk} f_{nj}] = \int_0^1 f_{nk}(\omega_1) \ f_{nj}(\omega_1)d\omega_1 = 0 \quad (k \neq j).$$

We will apply the Rademacher functions $\{r_n (\omega_2): n = 1,2,\ldots\}$ defined by

$$r_n(\omega_2): = \sum_{k=1}^{2^n} (-1)^{k-1} I_{[(k-1)/2^n, \ k/2^n)}(\omega_2), \qquad \omega_2 \in [0,1).$$

Obviously, $\{r_n\}$ is a sequence of independent, identically distributed r.v.'s with

$$(1.20) \qquad E[r_n] = 0 \quad \text{and} \quad E[r_n^2] = 1 \quad (n = 1,2,\ldots).$$

The role of the Rademacher functions in the theory of Banach spaces is well-known (see, e.g., Schwartz (1981)).

Finally, we set

$$X_{nk}(\omega_1,\omega_2): = f_{nk}(\omega_1)r_n(\omega_2) \quad (k = 1,2,\ldots,n; \ n = 1,2,\ldots).$$

By construction, $\{f_{nk}\}$ and $\{r_n\}$ are independent of each other. Thus, by (1.18) - (1.20),

$$E[X_{nk}] = E[f_{nk}] \ E[r_n] = 0,$$

$$E[X_{nk}^2] = E[f_{nk}^2] \ E[r_n^2] = \sigma_{nk}^2,$$

$$E[X_{nk}X_{nj}] = E[f_{nk}f_{nj}] \ E[r_n^2] = 0 \quad (k \neq j),$$

$$E[X_{nk}X_{mj}] = E[f_{nk}f_{mj}] \ E[r_n r_m] = 0 \quad (n \neq m),$$

which in turn provide relations (1.11) - (1.13).

By (1.15) and (1.16), each $\omega_1 \in [0,1)$ is in J_{nk} for infinitely many n, but for exactly one k when it is. Thus, for each $(\omega_1, \omega_2) \in I^2$

$$|\zeta_n| = |\frac{1}{n^\alpha} \sum_{k=1}^{n} X_{nk} (\omega_1, \omega_2)|$$

$$= \frac{1}{n^\alpha} | \frac{n^\alpha}{\epsilon_n} r_n(\omega_2)| = \frac{1}{\epsilon_n} \quad \text{infinitely often,}$$

whence (1.14) follows.

§2. Extensions

A) We can consider the following generalized array of r.v.'s

$$X_{11}, \quad X_{12}, \quad \ldots, \quad X_{1p_1}$$

$$X_{21}, \quad X_{22}, \quad \ldots, \quad X_{2p_2}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$X_{n1}, \quad X_{n2}, \quad \ldots, \quad X_{np_2}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

where $(p_n: n = 1,2,\ldots)$ is a not necessarily increasing sequence of positive integers. In case $p_n = n$ for all n, we get an ordinary (triangular) array.

B) We can substitute any sequence $(\lambda(n): n = 1,2,\ldots)$ of positive numbers for n^α in definition (1.3).

C) We can require only quasi-orthogonality (see, e.g. Móricz (1977)) instead of orthogonality. More exactly, we assume the fulfillment of (1.1), but instead of (1.2) we only need the existence of a generalized array $\{\rho_{nk}: k = 1,2,\ldots,p_n; n = 1,2,\ldots\}$ of nonnegative numbers such that

$$(2.1) \qquad \sum_{k=1}^{p_n} \rho_{nk} \leq C \qquad\qquad (n = 1,2,\ldots)$$

where the constant C does not depend on n, and

$$(2.2) \qquad |E[(X_{nk}, X_{nj})]| \leq \rho_{n,|k-j|+1} \sigma_{nk} \sigma_{nj}$$

$$(k,j = 1,2,\ldots,p_n; n = 1,2,\ldots).$$

In the particular case when $\rho_{n1} = 1$ and $\rho_{nk} = 0$ otherwise, we get ordinary orthogonality.

It is known (see again Móricz (1977)) that (2.1) and (2.2) imply that

$$E\left[\left\|\sum_{k=1}^{p_n} X_{nk}\right\|^2\right] \leq (1 + 2C) \sum_{k=1}^{p_n} \sigma_{nk}^2 \qquad (n = 1,2,\ldots).$$

Now the fulfillment of this moment inequality is crucial in the proof of Theorem 1.

To sum up, the following theorem can be proved along the same lines as Theorem 1.

Theorem 1A. <u>Let</u> $\{X_{nk}: k = 1,2,\ldots,p_n; n = 1,2,\ldots\}$ <u>be a generalized array in a separable Hilbert space</u> H <u>satisfying conditions</u> (1.1), (2.1) <u>and</u> (2.2), <u>and let</u> $(\lambda(n): n = 1,2,\ldots)$ <u>be a sequence of positive numbers.</u> <u>If</u>

$$(2.3) \qquad \sum_{n=1}^{\infty} \frac{1}{\lambda^2(n)} \sum_{k=1}^{P_n} \sigma_{nk}^2 < \infty,$$

then

$$(2.4) \qquad \lim_{n\to\infty} \frac{1}{\lambda(n)} \sum_{k=1}^{P_n} X_{nk} = 0 \qquad \underline{completely}.$$

Even in this very general setting, condition (2.3) is the best possible one to ensure (2.4). The way we proved Theorem 2 makes it possible to prove the following more general theorem.

Theorem 2A. <u>Let</u> $\{\sigma_{nk}: k = 1,2,\ldots,p_n; n = 1,2,\ldots\}$ <u>be a generalized array of nonnegative numbers such that</u>

$$\sum_{n=1}^{\infty} \frac{1}{\lambda^2(n)} \sum_{k=1}^{P_n} \sigma_{nk}^2 = \infty.$$

<u>Then there exists a generalized array</u> $\{X_{nk}\}$ <u>of real valued r.v.'s such that conditions</u> (1.11) <u>and</u> (2.12) <u>are satisfied</u>,

$$E[X_{nk}X_{mj}] = 0 \qquad (n \neq m \text{ or } k \neq j;$$

$$k = 1,2,\ldots,p_n; \ j = 1,2,\ldots,p_m; \ n,m = 1,2,\ldots),$$

and

$$\limsup_{n\to\infty} \left|\frac{1}{\lambda(n)} \sum_{k=1}^{P_n} X_{nk}\right| = \infty \qquad \underline{pointwise}.$$

§3. Banach Space Valued Random Variables

Many authors have contributed to the development of the theory of Banach space valued r.v.'s. However, we will only need reference to the

type p spaces of Hoffmann-Jørgensen and Pisier (1976), and the orthogonality moment inequalities of Howell and Taylor (1981). Additional details on basic properties and measurability considerations are available in the literature cited there.

Orthogonality in the general Banach space situation becomes much more problematic as the following example illustrates.

Example 1. Let $1 \leq q < \infty$ and ℓ^q denote the set of sequences $(x_k: k = 1,2,...)$ of real numbers such that

$$\|(x_k)\| := \left(\sum_{k=1}^{\infty} |x_k|^q \right)^{1/q} < \infty.$$

It is well-known that ℓ^q with the usual vector operation and the norm $\|\cdot\|$ defined above is a separable Banach space. Setting

$$X_{nk} := (0,...,0,r_k,0,...) \quad (k = 1,2,...,n; \; n = 1,2,...)$$

where the $\{r_k\}$ are the Rademacher functions, we get an array $\{X_{nk}\}$ consisting of rowwise independent, zero mean r.v.'s. So, they should be rowwwise orthogonal with respect to any reasonable definition of orthogonality. However, in trying to develop (1.6), we observe that

$$E\left[\| \sum_{k=1}^{n} X_{nk} \|^2 \right] = E\left[\left(\sum_{k=1}^{n} |r_k|^q \right)^{2/q} \right] = n^{2/q}$$

while

$$\sum_{k=1}^{n} E\left[\|X_{nk}\|^2 \right] = n.$$

Thus, for $1 \leq q < 2$ the inequality

$$E\left[\| \frac{1}{n^{2\alpha}} \sum_{k=1}^{n} X_{nk} \|^2 \right] = \frac{n^{2/q}}{n^{2\alpha}} > \frac{n}{n^{2\alpha}} = \frac{1}{n^{2\alpha}} \sum_{k=1}^{n} E\left[\|X_{nk}\|^2 \right]$$

goes in the wrong direction, and thus we are unable to duplicate the results of Section 1.

Hoffmann-Jørgensen and Pisier (1976) addressed this problem by defining a separable Banach space B to be _of type_ p, $1 \le p \le 2$, if for every sequence $\{X_k: k = 1,2,...\}$ of independent, zero mean r.v.'s in B with

$$(3.1) \qquad E[\|X_k\|^p] < \infty,$$

and for every $n \ge 1$, we have

$$(3.2) \qquad E[\|\sum_{k=1}^{n} X_k\|^p] \le C \sum_{k=1}^{n} E[\|X_k\|^p]$$

where C is a constant not depending on n and $\{X_k\}$.

Clearly, every separable Banach space is of type 1 and every separable Hilbert space is of type 2 even with equality holding in (3.2) for $C = 1$. Moreover, the ℓ^p spaces are at most of type min $(2,p)$, $1 \le p < \infty$.

Howell and Taylor (1981) used _James type orthogonality_ in defining a sequence $\{X_k\}$ of r.v.'s in a separable Banach space B to be orthogonal in $L^p(B)$, $1 \le p < \infty$, if (3.1) is satisfied for all k and

$$(3.3) \qquad E[\|\sum_{k=1}^{n} a_{\pi(k)} X_{\pi(k)}\|^p] \le E[\|\sum_{k=1}^{n+m} a_{\pi(\epsilon)} X_{\pi(k)}\|^p]$$

for all sequences $\{a_k\}$ of real numbers, for all permutations π of the positive integers $\{1,2,...,m+n\}$, and for all n and m.

It is clear that orthogonal r.v.'s in a separable Hilbert space satisfy (3.1) and (3.3) with $p = 2$. It is not hard to see that the

r.v.'s defined in ℓ^q in Example 1 satisfy (3.1) and (3.3) for any p, $1 \leq p \leq 2$. Note, q need not equal p.

Remark 2. Proposition 2.1 of Howell and Taylor (1981) states that a separable Banach space B is of type p if and only if (3.2) is satisfied for all n and for all sequences $\{X_k\}$ which are orthogonal in $L^p(B)$.

Inequality (3.2) was actually established for the weaker concept of unconditional semi-basic (ucsb) r.v.'s but in this case the constant C depends on the particular ucsb sequence $\{X_k\}$. On the other hand, in the case of orthogonal r.v.'s the constant C depends only on the space B.

To illustrate the generality of the concept of orthogonality in Banach spaces we present the following

Example 2. Let $1 \leq p < 2$ and let Y be a Borel measurable, real valued function defined on [0,1) such that

$$E[\,|Y|^p\,] = \int_0^1 |Y(\omega_1)|^p \, d\omega_1 < \infty$$

but

$$E[\,Y^2\,] = \int_0^1 Y^2(\omega_1)d\omega_1 = \infty.$$

Define for $(\omega_1, \omega_2) \in [0,1) \times [0,1) = I^2$

$$X_n(\omega_1, \omega_2): = Y(\omega_1) \, r_n(\omega_2) \qquad (n = 1,2,\ldots)$$

where the $\{r_n\}$ are the Rademacher functions.

Then the r.v.'s $\{X_n\}$ are not orthogonal in the classical sense since $E[X_n X_m]$ is not defined. Indeed,

$$E[\,|X_n X_m|\,] = E[\,Y^2|r_n r_m|\,] = E[\,Y^2\,] = \infty.$$

On the other hand, the r.v.'s $\{X_n\}$ are James type orthogonal in $L^p(I^2)$. This is a consequence of the fact that the r.v.'s $\{r_n\}$ are independent and have zero mean, and therefore

$$E\left[\|\sum_{k=1}^{n} a_{\pi(k)} X_{\pi(k)}\|^p\right]$$

$$= E\left[|Y|^p\right] E\left[\|\sum_{k=1}^{n} a_{\pi(k)} r_{\pi(k)}\|^p\right]$$

$$\leq E\left[|Y|^p\right] E\left[\|\sum_{k=1}^{n+m} a_{\pi(k)} r_{\pi(k)}\|^p\right]$$

$$= E\left[\sum_{k=1}^{n+m} a_{\pi(k)} X_{\pi(k)}\|^p\right].$$

Now if $\{X_{nk}: k = 1,2,\ldots,n; n = 1,2,\ldots\}$ is an array of rowwise orthogonal (in $L^p(B)$) r.v.'s in a Banach space B of type p, then it follows from (3.2) that

$$\sum_{n=1}^{\infty} E\left[\|\frac{1}{n^\alpha} \sum_{k=1}^{n} X_{nk}\|^p\right] \leq \sum_{n=1}^{\infty} \frac{C}{n^{p\alpha}} \sum_{k=1}^{n} E\left[\|X_{nk}\|^p\right]$$

and the following form of Theorem 1 is obtained.

Theorem 1B. <u>Let</u> $\{X_{nk}\}$ <u>be an array of rowwise orthogonal</u> (<u>in</u> $L^p(B)$) <u>r.v.'s in a Banach space</u> B <u>of type</u> p <u>for some</u> $1 \leq p \leq 2$. <u>If</u>

$$(3.4) \qquad \sum_{n=1}^{\infty} \frac{1}{n^{p\alpha}} \sum_{k=1}^{n} E\left[\|X_{nk}\|^p\right] < \infty$$

<u>for some</u> $\alpha > 0$, <u>then we have</u> (1.5).

Remark 3. Since the real line R is of type p for each $1 \leq p \leq 2$, Theorem 1B extends both the concept of orthogonality for real valued r.v.'s

and the range of applicable moment conditions which yield (1.5).

Sufficient condition (3.4) is also best possible even for real valued r.v.'s as the next theorem shows.

Theorem 2B. <u>Let</u> $\{\tau_{nk}\}$ <u>be an array of nonnegative real numbers such that</u>

$$\sum_{n=1}^{\infty} \frac{1}{n^{p\alpha}} \sum_{k=1}^{n} \tau_{nk}^{p} = \infty$$

<u>for some</u> $1 \le p \le 2$ <u>and some</u> $\alpha > 0$. <u>Then there exists an array</u> $\{X_{nk}\}$ <u>of real valued r.v.'s such that conditions</u>. (1.11) <u>and</u>

$$E[\,|X_{nk}|^{p}\,] = \tau_{nk}^{p}$$

<u>are satisfied, they are rowwise orthogonal in</u> $L^{p}(I^2)$ <u>and we have</u> (1.14).

For the proof of Theorem 2B, replace (1.15) by

$$c_n := \frac{\varepsilon_n^p}{n^{p\alpha}} \sum_{j=1}^{n} \tau_{nj}^{p} \le 1 \qquad (n = 1,2,\ldots)$$

and follow the steps in the proof of Theorem 2 with $d_{nk} := \sum_{i=1}^{n-1} c_i + \frac{\varepsilon_n^p}{n^{p\alpha}} \sum_{j=1}^{k} \tau_{nj}^{p}$. The rowwise orthogonality in $L^p(I^2)$ follows since $J_{n1}, \ldots, J_{nn}$ are disjoint subsets. It also follows that $X_{n1}, \ldots, X_{nn}$ are orthogonal in the usual sense. Banach space versions of Corollary 1 and Theorems 1A and 1B are also available with similar conditions.

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